**Driven Oscillations** (linear damping)

- Damped harmonic oscillator driven by time dependent external force:

\[ m\ddot{x} + c\dot{x} + kx = F_{\text{ext}}(t) \]

  inhomogeneous!

- Look at simple, periodic driving force:

\[ F_{\text{ext}}(t) = F_0 \cos(\omega t) = \text{Re}\{F_0 e^{i\omega t}\} \]

  it will be simpler to deal with complex numbers

- Solution to differential equation will be the sum of two parts:

\[ x(t) = x_c(t) + x_p(t) \]

  complementary soln.
  (soln to homogeneous eqn)
  transient term
  (we know this already!)

  "particular integral"
  (soln to inhomogeneous bit)
  steady-state term
Driven Oscillations (linear damping)

\[ x(t) = x_c(t) + x_p(t) = C_1 e^{-\gamma t} e^{i \omega_0 t} + C_2 e^{-\gamma t} e^{-i \omega_0 t} + A e^{i(\omega t - \delta)} \]

or

\[ x(t) = x_c(t) + x_p(t) = A d e^{-\gamma t} \cos(\omega_d t - \phi) + A \cos(\omega t - \delta) \]

where

\[ \delta = \tan^{-1}\left( \frac{2 \gamma \omega}{\omega_0^2 - \omega^2} \right) \]

\[ A = \frac{F_0}{m} \sqrt{\left( \omega_0^2 - \omega^2 \right)^2 + 4 \gamma^2 \omega^2} \]

where

\[ \omega^2 = \omega_0^2 - 2 \gamma^2 \]

resonance

\[ \gamma = \frac{1}{2L} \int_{-L}^{L} f(x') dx' \]

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x') \cos\left( \frac{n \pi x'}{L} \right) dx' \]

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x') \sin\left( \frac{n \pi x'}{L} \right) dx' \]

Similarly, the function is instead defined on the interval \([0, L]\), the above equations simply become

\[ a_0 = \frac{1}{L} \int_{0}^{L} f(x') dx' \]

\[ a_n = \frac{1}{L} \int_{0}^{L} f(x') \cos\left( \frac{n \pi x'}{L} \right) dx' \]

\[ b_n = \frac{1}{L} \int_{0}^{L} f(x') \sin\left( \frac{n \pi x'}{L} \right) dx' \]

Fourier Series

For a function defined on the interval \([-L, L]\), where

\[ f(x') = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left( \frac{n \pi x'}{L} \right) + \sum_{n=1}^{\infty} b_n \sin\left( \frac{n \pi x'}{L} \right) \]

where

\[ a_0 = \frac{1}{L} \int_{-L}^{L} f(x') dx' \]

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x') \cos\left( \frac{n \pi x'}{L} \right) dx' \]

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x') \sin\left( \frac{n \pi x'}{L} \right) dx' \]
Hamilton’s Principle

• Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.

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