Driven Oscillations (linear damping)

• Damped harmonic oscillator driven by time dependent external force:

\[ m\ddot{x} + c\dot{x} + kx = F_{\text{ext}}(t) \]

inhomogeneous!

• Look at simple, periodic driving force:

\[ F_{\text{ext}}(t) = F_0 \cos(\omega t) = \text{Re}\{F_0 e^{i\omega t}\} \]

it will be simpler to deal with complex numbers

• Solution to differential equation will be the sum of two parts:

\[ x(t) = x_c(t) + x_p(t) \]

complementary soln. (soln to homogeneous eqn)

transient term

"particular integral" (soln to inhomogeneous bit)

steady-state term

{we know this already!}
Driven Oscillations (linear damping)

\[ x(t) = x_c(t) + x_p(t) \]

\[ x_c(t) = C_x e^{-\gamma t} e^{i\omega t} + C_x e^{-\gamma t} e^{-i\omega t} = Ae^{-\gamma t} \cos(\omega_d t - \delta) \]

how about a trial solution of \( x_p(t) = Ae^{i(\omega_d t - \delta)} \)?

- This works if ......

\[ \delta = \tan^{-1}\left( \frac{2\gamma \omega}{\omega_o^2 - \omega^2} \right) = \tan^{-1}\left( \frac{c \omega}{k - m \omega^2} \right) \]

\[ A = \frac{F_o}{\left[ (k - m \omega^2)^2 + c^2 \omega^2 \right]^{1/2}} = \frac{F_o / m}{\left[ \left( \omega_o^2 - \omega^2 \right)^2 + 4\gamma^2 \omega^2 \right]^{1/2}} = \frac{F_o}{m D(\omega)} \]

\[ \omega_o^2 = \omega_o^2 - 2\gamma^2 \]

Driven Oscillations (linear damping)

\[ x(t) = x_c(t) + x_p(t) = C_x e^{-\gamma t} e^{i\omega_d t} + C_x e^{-\gamma t} e^{-i\omega_d t} + Ae^{i(\omega_d t - \delta)} \]

or

\[ x(t) = x_c(t) + x_p(t) = A_x e^{-\gamma t} \cos(\omega_d t - \phi) + A \cos(\omega t - \delta) \]

where

\[ \delta = \tan^{-1}\left( \frac{2\gamma \omega}{\omega_o^2 - \omega^2} \right) \]

\[ A = \frac{F_o / m}{\left[ (\omega_o^2 - \omega^2)^2 + 4\gamma^2 \omega^2 \right]^{1/2}} \]
Fourier Series

For a function defined on the interval \([-L, L]\), where

\[
f(x') = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n \pi x'}{L} \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{n \pi x'}{L} \right)
\]

where

\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x') \, dx'
\]
\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x') \cos \left( \frac{n \pi x'}{L} \right) \, dx'
\]
\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x') \sin \left( \frac{n \pi x'}{L} \right) \, dx'.
\]

Similarly, if the function is instead defined on the interval \([0, L]\), the above equations simply become

\[
a_0 = \frac{1}{L} \int_{0}^{L} f(x') \, dx'
\]
\[
a_n = \frac{1}{L} \int_{0}^{L} f(x') \cos \left( \frac{n \pi x'}{L} \right) \, dx'
\]
\[
b_n = \frac{1}{L} \int_{0}^{L} f(x') \sin \left( \frac{n \pi x'}{L} \right) \, dx'.
\]