Simple Harmonic Oscillations (no damping)

Energy considerations  \[ x(t) = A \cos(\omega t - \phi) \]

\[ E_{\text{mech}}(t) = \frac{1}{2} m x^2 + \frac{1}{2} k x^2 = \frac{1}{2} k A^2 \]

\[ \omega = \sqrt{\frac{k}{m}} \]
**Simple Harmonic Oscillations** (no damping)

Energy considerations for a particle oscillating about a point of stable equilibrium.

Any potential well can be modeled as approximately parabolic for small enough oscillations.

- Can do a Taylor series expansion about equilibrium position, $x_0$:
  \[ U(x) = U(x_0) + \left. \frac{dU(x)}{dx} \right|_{x_0} (x-x_0) + \frac{1}{2!} \left. \frac{d^2U(x)}{dx^2} \right|_{x_0} (x-x_0)^2 + \ldots \]

- with $u = x - x_0$ and with $u < 1$, we can write
  \[ \omega = \sqrt{\frac{k}{m}} \]

In summary: The Gradient

- Del acting on a scalar is called the gradient
  \[ \mathbf{\nabla} P(x, y, z) \equiv \mathbf{i} \frac{\partial P}{\partial x} + \mathbf{j} \frac{\partial P}{\partial y} + \mathbf{k} \frac{\partial P}{\partial z} \]

- $\mathbf{\nabla} P(x, y, z)$ points in the direction of maximum increase in the function $P(x, y, z)$

- $|\mathbf{\nabla} P(x, y, z)|$ gives the “slope” (rate of increase) along this maximal direction
Differential Calculus: Del on vectors?

• The del operator is a differential operator that “acts on,” rather than “multiplies” the function to its right.
• Acting on vectors, we have two options of interest:

\[ \nabla \cdot \vec{V} \quad \text{The Divergence} \\
\nabla \times \vec{V} \quad \text{The Curl} \]

**NOTE:** like the gradient, these are COORDINATE SYSTEM DEPENDENT!!

Differential Calculus: The Divergence

• Gives feel for how much the field is spreading out or DIVERGING from the point in question!
• Often associate with sources or sinks of the field.
• Sort of a “slope of the components.”
• Results in a scalar function!

\[ \nabla \cdot \vec{V} = i \frac{\partial V_x}{\partial x} + j \frac{\partial V_y}{\partial y} + k \frac{\partial V_z}{\partial z} \]

The Divergence in Cartesian coords.

[COORDINATE SYSTEM DEPENDENT!!]
Differential Calculus: The Curl

• Gives feel for how much the vector field is rotating or CURLING about the point in question!

• Results in a vector function.

\[ \nabla \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} \]

or

\[ \nabla \times \vec{B} = \hat{i}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{j}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{k}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) \]

[COORDINATE SYSTEM DEPENDENT!!]

The Curl in Cartesian coords.

A Theorem

“Curless” or irrotational vector functions

i) \( \nabla \times \vec{F} = 0 \quad \forall \text{ space} \)

ii) \( \int_{a}^{b} \vec{F} \cdot d\vec{l} \) is path independent

iii) \( \oint \vec{F} \cdot d\vec{l} = 0 \quad \forall \text{ closed loops} \)

iv) \( \vec{F} = \nabla V \)

Conservative forces are examples of such functions
Work, Force, and Potential Energy

\[ W = \oint_{\vec{r}_f} \vec{F} \cdot d\vec{r} = -\Delta U = -(U_f - U_i) \]

where \( \vec{F} = -\nabla U \) and such forces are conservative

Work and Kinetic Energy

\[ W_{net} = \oint_{\vec{r}_f} \vec{F}_{net} \cdot d\vec{r} = \Delta K = (K_f - K_i) \]

Conservative Forces

- Conservative forces can be written as the gradient of some scalar function:

\[ \vec{F} = -\nabla U = -\left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) U = -\hat{i} \frac{\partial U}{\partial x} - \hat{j} \frac{\partial U}{\partial y} - \hat{k} \frac{\partial U}{\partial z} \]

- One can show that this is equivalent to any of the following:

\[ \nabla \times \vec{F} = 0 \quad \oint \vec{F} \cdot d\vec{r} = 0 \]

\[ \oint_{\vec{r}_f} \vec{F} \cdot d\vec{r} = \text{path independent} \]
The Simple Pendulum

• Relative to the lowest point in the swing of the bob: \( U(\vec{r}) = mgL(1 - \cos \theta) \)

\[
\vec{F} = -\nabla U = -\vec{r} \frac{\partial U}{\partial \vec{r}} - \dot{\theta} \frac{1}{L} \frac{\partial U}{\partial \theta} - k \frac{\partial U}{\partial z} = \left( -\dot{\theta} m g \sin \theta \right)
\]

Note that \( \nabla \) operator takes on different forms in different coordinate systems (see inside back cover of textbook!); here, \( r = L \)

\[
\vec{F} = m \ddot{r} = m(\ddot{r} - r \dot{\theta}^2) + \dot{\theta} (r \ddot{\theta} + 2r \dot{\theta}) + k(\dot{z}) = \left( m f(\theta) - \dot{\theta} m g \sin \theta \right)
\]

this centripetal term has no potential associated with it

The Simple Pendulum

• The angular component of the equation of motion is then

\[
mL \ddot{\theta} = -m g \sin \theta
\]

\[
\therefore \dot{\theta} = -\left( \frac{g}{L} \right) \sin \theta
\]

• For small angles, \( \dot{\theta} = -\left( \frac{g}{L} \right) \theta \)

We identify \( \omega^2 = \left( \frac{g}{L} \right) \)

Can also use:

\[
\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1}{m} \left. \frac{d^2 U(\chi)}{dx^2} \right|_{\chi=\theta}}
\]
The Physical Pendulum

- Another approach:
  \[ \ddot{x} = \frac{I}{\ell} \Rightarrow \ell \ddot{\theta} = -mgL \sin \theta \]
  \[ \therefore \dot{\theta} = -\left(\frac{mgL}{I}\right) \sin \theta \]
  
- For small angles
  \[ \dot{\theta} = -\left(\frac{mgL}{I}\right) \theta \quad \text{We identify} \quad \omega_0^2 = \left(\frac{mgL}{I}\right) \]

Damped Oscillations (linear damping)

Energy considerations \[ x(t) = Ae^{-\gamma t} \cos(\omega_d t - \phi) \]

Mechanical energy dissipated as frictional heat

\[ \omega_d^2 = \omega_0^2 - \gamma^2 \]
\[ \omega_0 = \sqrt{\frac{k}{m}} \]
\[ \gamma = \frac{c}{2m} \]